UNIFORM STRUCTURES AND SQUARE ROOTS IN TOPOLOGICAL GROUPS

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PART II.

ABSTRACT

Commutative groups uniformly homeomorphic to certain Banach spaces are considered. Results on the relation between the structure of the topological group and the Banach space are obtained.

In this part of the paper we turn from the abstract approach to Hilbert's fifth problem to a more concrete approach and so we first give some necessary geometrical results.

4. Roundness and uniform convexity

4.1. We recall a definition from Enflo [2]: A metric space is said to have roundness p if p is the supremum of the set of q : s with the property: for every quadruple of points $a_{00}, a_{01}, a_{11}, a_{10}$ we have

$$\begin{aligned}[d(a_{00}, a_{01})]^q + [d(a_{01}, a_{11})]^q + [d(a_{11}, a_{10})]^q + [d(a_{10}, a_{00})]^q \\ \geqq [d(a_{00}, a_{11})]^q + [d(a_{01}, a_{10})]^q.\end{aligned} \tag{1}$$

The triangle inequality shows that (1) is always satisfied for $q = 1$. If the metric space has the property that some pair of points has a metric middle point then (1) is not satisfied for all quadruples if $q > 2$. We see this by choosing a_{01} to be the middle point between a_{00} and a_{11} and putting $a_{10} = a_{01}$. The propositions of this section show that the concept "roundness" is quite natural especially for Banach spaces. Let $q > 1$.

PROPOSITION 4.1.1. *If in a Banach space* (1) *holds with some exponent q for all quadruples where $a_{00}, a_{01}, a_{11}, a_{10}$ are corners in a parallellogram with (a_{00}, a_{11}) as one diagonal, then* (1) *holds with the exponent q for all quadruples in the Banach space.*

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PROOF. It is enough to prove that if the inequality holds with $a_{00} = 0$, $a_{01} = (x-y)/2$, $a_{11} = x$ and $a_{10} = (x+y)/2$, then it holds with $a_{00} = 0$, $a_{01} = x-y-b$, $a_{11} = x$ and $a_{10} = x - b$ for every b, that is if $2\cdot \|(x-y)/2\|^q + 2\cdot \|(x+y)/2\|^q \geqq \|x\|^q + \|y\|^q$ then $\|b\|^q + \|x-y-b\|^q + \|x-b\|^q + \|y+b\|^q \geqq \|x\|^q + \|y\|^q$ for every b. If r_1 and r_2 are positive real numbers and $r_1 + r_2$ is assumed to be constant then $r_1^q + r_2^q$ attains its minimum when $r_1 = r_2$. This and the triangle inequality gives that $\|b\|^q + \|x-y-b\|^q$ attains its minimum when $b = (x-y)/2$. For this, b, $\|x-b\|^q + \|y+b\|^q$ also attains its minimum. This proves the proposition.

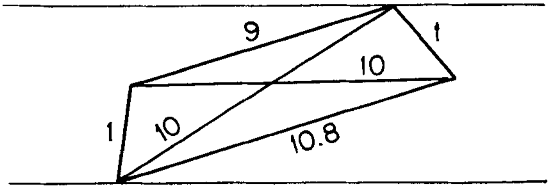
LEMMA. *If $r_1, r_2, \cdots, r_6$ are positive real numbers and $\max(r_1,r_2,r_3,r_4) \leqq \max(r_5,r_6)$ and $r_1^q + r_2^q + r_3^q + r_4^q = r_5^q + r_6^q$, then $r_1^p + r_2^p + r_3^p + r_4^p < r_5^p + r_6^p$ if $p > q$.*

PROOF. We form $d/dp(r_1^p + r_2^p + r_3^p + r_4^p - r_5^p - r_6^p) = p^{-1}(r_1^p \cdot \log r_1^p + \cdots + r_4^p \cdot \log r_4^p - r_5^p \cdot \log r_5^p - r_6^p \cdot \log r_6^p)$. If a and b are positive real numbers and $a + b$ is assumed to be constant then $a \cdot \log a + b \cdot \log b$ increases with $|a-b|$. Thus it is enough to prove $d/dp(p \geqq q) < 0$ when we have minimized $r_5^p \cdot \log r_5^p + r_6^p \cdot \log r_6^p$ under the assumptions that $r_5^p + r_6^p$ is constant and $\max(r_1,r_2,r_3,r_4) \leqq \max(r_5,r_6)$. Thus it is enough to prove $d/dp(p \geqq q) < 0$ if $\max(r_1,r_2,r_3,r_4) = \max(r_5,r_6)$ or if $\max(r_1,r_2,r_3,r_4) < \min(r_5,r_6)$ and both these cases are easy.

PROPOSITION 4.1.2. *If (1) holds with the exponent q for all quadruples in a Banach space, then (1) holds also with the exponent q_1 if $1 \leqq q_1 \leqq q$.*

PROOF. Since in a Banach space the largest edge in a parallellogram is not larger than the largest diagonal the preceding lemma shows that (1) holds for all parallellograms with the exponent q_1, $1 \leqq q_1 \leqq q$. And so Proposition 4.1.1 shows that (1) holds with the exponent q_1 for all quadruples in the Banach space.

REMARK. Proposition 4.1.2 is not true for general metric spaces which can be seen by the following example:



In this metric space we have $10{,}8^2 + 1^2 + 1^2 + 9^2 < 10^2 + 10^2$ but (1) holds with the exponent q for all quadruples (also for those quadruples in which some points are equal) if q is the smallest number > 2 such that $10{,}8^q + 1^q + 1^q + 9^q = 10^q + 10^q$.

In the following chapters the concepts "roundness > 1" and "uniform convexity" will both be of importance and so we have to investigate if there is a simple connection between them. The following propositions show that there is no such simple connection. We recall that a Banach space is said to be uniformly convex if there is a function $\delta(\varepsilon) > 0$ for $\varepsilon > 0$ such that $\| x + y \| \leqq 2 \cdot (1 - \delta(\varepsilon))$ when $\| x \| \leqq 1$, $\| y \| \leqq 1$, $\| x - y \| \geqq \varepsilon$.

PROPOSITION 4.1.3. *If in a Banach space B, the unit sphere of some two-dimensional subspace of B has a corner, then B has roundness* 1.

PROOF. Let C be the two-dimensional subspace whose unit sphere has a corner x_0. Let x_n and y_n be two sequences of points on the unit sphere of C which converge to x_0 from different sides and which satisfy $\| x_n - x_0 \| = \| y_n - x_0 \|$. Consider the parallellograms with corners in $x_n, y_n, -x_n, -y_n$ and with $(x_n, -x_n)$ as one diagonal. In this parallelogram both diagonals have length 2 and since the unit sphere has a corner in x_0 we have $\| x_n + y_n \| + K_n \cdot \| x_n - y_n \| = 2$, where K_n is a sequence bounded away from 0. Since $\| x_n - y_n \| \to 0$ as $n \to \infty$ this equality gives that if $q > 1$ then $2 \cdot \| x_n + y_n \|^q + 2 \cdot \| x_n - y_n \|^q < 2 \cdot 2^q$, if n is sufficiently large.

Proposition 4.1.3 shows that there exist uniformly convex Banach spaces which have roundness 1. If B is a Banach space, let p_B denote the roundness of B.

PROPOSITION 4.1.4. *If the boundary of the unit sphere of a two-dimensional Banach space is a regular C_2-curve, then $p_B > 1$.*

PROOF. If the boundary of the unit sphere is a regular C_2-curve, then it is easy to verify that $x \to \| x \|$ is twice continuously differentiable except at 0. Let x be a point with $\| x \| = 1$ and consider a parallellogram with corners in $0, x, x + ty$ and ty where $\| y \| = 1$. Then $\| x + ty \| = 1 + K_1 \cdot t + 0(t^2)$ and $\| x - ty \| = 1 - K_1 \cdot t + 0(t^2)$ where $0(t^2)$ holds uniformly on $\| x \| = 1$ as $t \to 0$. This gives that if $p_B = 1$ and there exists a sequence of non-degenerated parallellograms with $\sum s_{\alpha,n}^{p_n} = \sum d_{\beta,n}^{p_n}$ (where $s_{\alpha,n}$ denotes the lengths of the edges and $d_{\beta,n}$ the lengths of the diagonals) where $p_n \to 1$ we must have $s_{n,max} \leqq K \cdot s_{n,min}$ for all n and some K. We now assume that there is such a sequence with $s_{n,max} \leqq K \cdot s_{n,min} = 1$. By choosing a convergent subsequence we get a parallellogram with $\Sigma s_\alpha = \Sigma d_\beta$. We can assume that the central point of this parallellogram is 0 and that the corners are $x + y, x - y$, $-x - y$ and $-x + y$. We can also assume that $\| x + y \| = 1$. Since we have by assumption $4 \| x \| + 4 \| y \| = 2 \| x + y \| + 2 \| x - y \|$ and by the triangle inquality $\| x \| + \| y \| \geqq \| x + y \|$ and $\| x \| + \| y \| \geqq \| x - y \|$ we have $\| x \| + \| y \| = \| x - y \| = 1$. Consider the line through $y/\| y \|$ and $x + y$. This line intersects the line through 0 and x at the point $x/(1 - \|y\|)$. Now $\| x/(1 - \| y \|)\| = 1$ and so the linear segment which connects $y/\| y \|$ and $x/(1 - \| y \|)$ lies entirely on the boundary of the unit sphere. This gives that the boundary of the unit sphere is the boundary of a parallellogram and so it is not a regular C_2-curve. This proves the proposition.

Proposition 4.1.4 gives that a Banach space with roundness > 1 need not be uniformly convex. The proof of proposition 4.1.4 also gives the following proposition which shows that the class of Banach spaces with roundness > 1 is fairly large. The complication that we cannot find a convergent subsequence of parallellograms in the second part of the proof is easy to handle.

PROPOSITION 4.1.5. *If in a Banach space B, $x \to \| x \|$ is twice continuously Frechet differentiable in some set $0 < r_1 \leqq x \leqq r_2$ and the second derivative is bounded in this set, then $p_B > 1$.*

In [2] it was proved that $L_p(0,1)$, $1 \leqq p \leqq 2$, has roundness p.

We shall say that a set of 2^n points (not necessarily different) in a metric space is an n-dimensional cube, if the points are indexed by the 2^n n-vectors whose components are 0 and 1. We shall say that a pair of points in an n-dimensional cube is an edge if the indexes of the points are different in exactly one component. We shall say that a pair of points is an m-diagonal if m components of the indexes are different. The importance of the concept roundness in this paper depends heavily on the following theorem which was proved in [2]: In an n-dimensional cube in a metric space with roundness p we have $n^{1/p} \cdot s_{max} \geqq d_{min}$ where s_{max} is the length of the largest edge and d_{min} is the length of the smallest n-diagonal. (2)

The following theorem is the weaker counterpart of (2) for uniformly convex Banach spaces.

THEOREM 4.1.6. *If B is a uniformly convex Banach space, then for every $K > 1$, there is a w, such that for all n and m with $n/m > w$ and every n-dimensional cube H_n in B there is an m_1-diagonal in H_n, $m_1 \geqq m$, with length $\leqq m_1/K \cdot s_{max}$.*

PROOF. The uniform convexity of B gives taat there exists a positive δ with the property that if the lengths of the two diagonals in a quadruple are > 1 and $> 1/K$ then there is an edge of length $> (1+\delta)/2$ in the quadruple. Let H_n be an n-dimensional cube in B, $n = 2^p$. First consider the quadruple of points where one diagonal d_{11} consists of the points $a_{00\ldots0}$ and $a_{11\ldots1}$ and the other diagonal d_{21} consists of the points $a_{\underbrace{00\ldots0}_{n/2\ 0:s}\underbrace{11\ldots1}_{n/2\ 1:s}}$ and $a_{\underbrace{11\ldots1}_{n/2\ 1:s}\underbrace{00\ldots0}_{n/2\ 0:s}}$. If d_{11} and d_{21} have both length $> n/K \cdot s_{max}$ then the length of some edge in this quadruple is $> (n/2K)(1+\delta)$. We can assume that the length of $(a_{\underbrace{11\ldots1}_{n/2\ 1:s}\underbrace{00\ldots0}_{n/2\ 0:s}}, a_{00\ldots0})$ is $> (n/2K)(1+\delta)$. Now consider the quadruple of points where one diagonal d_{12} consists of the points $a_{\underbrace{11\ldots1}_{n/2\ 1:s}\underbrace{00\ldots0}_{n/2\ 0:s}}, a_{00\ldots0}$ and the other diagonal d_{22} consists of the points $a_{\underbrace{11\ldots1}_{n/4\ 1:s}\underbrace{00\ldots0}_{3n/4\ 0:s}}, a_{\underbrace{00\ldots0}_{n/4\ 0:s}\underbrace{11\ldots1}_{n/4\ 1:s}\underbrace{00\ldots0}_{n/2\ 0:s}}$. Now if d_{22} has lengt $> n/2K \cdot s_{max}$ then the length of the largest edge in this quadruple is $(n/4K)(1+\delta)^2$. If n is sufficiently large we can continue this process r times where r is the smallest number such that $(1+\delta)^r > K$. Some of these times we will find an m_1-diagonal with length $\leqq m_1/K \cdot s_{max}$ for otherwise we would have an $n/2^r$ diagonal with length $> (n/K \cdot 2^r) \cdot (1+\delta)^r \cdot s_{max}$, which contradicts the triangle inequality. If n is not of the form 2^p we can consider a 2^p-dimensional subcube of H_n, where p is the largest integer such that $2^p < n$, and so the theorem is proved also for this case.

5. Existence and largeness of square roots in groups

5.1. Existence and largeness of square roots in Banach groups. We shall say that B is a commutative Banach group if (1) B is a Banach space, (2) there is defined on B an operation $(x, y) \to xy$ which makes B a commutative topological group with 0 as unit element. We shall say that U is a commutative local Banach group if (1) U is an open neighbourhood of 0 in a Banach space, (2) there is defined on some open neighbourhood of 0 an operation $(x, y) \to xy$ which makes U a commutative local group with 0 as unit element.

THEOREM 5.1.1. *If in a commutative Banach group B $\| x_2 y - x_1 y \| = o(\| x_2 - x_1 \|^{1/p_B})$ uniformly in x_1, x_2 and y as $\| x_2 - x_1 \| \to 0$, then the set of elements of the form $z = y^2$ is dense in B.*

PROOF. Consider an element $z \in B$, let n be an integer and form the elements $y_m = (mz/n) \cdot ((m-1)z/n)^{-1}$, $1 \leqq m \leqq n$. Then $y_1 y_2 \cdots y_n = z$. Now consider the set of 2^n points which we get by forming all products $0 \cdot y_{k_1} y_{k_2} \cdots y_{k_j}$ where in every product y_v appears once or not at all. This set becomes an n-dimensional cube if for every point in the set we let the m : *th* component of the n-vector be 1 if y_m appears in the product and 0 otherwise. In this cube we have

$$s_{max} = \max \| y_v y_{k_1} y_{k_2} \cdots y_{k_j} - y_{k_1} y_{k_2} \cdots y_{k_j} \|$$

$$= \max \left\| \frac{vz}{n} \cdot \left(\frac{(v-1)z}{n} \right)^{-1} \cdot y_{k_1} y_{k_2} \cdots y_{k_j} - \frac{(v-1)z}{n} \cdot \left(\frac{(v-1)z}{n} \right)^{-1} y_{k_1} y_{k_2} \cdots y_{k_j} \right\|$$

$= o(\| z/n \|^{1/p_B})$ as $n \to \infty$ by assumption. If (b_D, b) is an n-diagonal in this cube we have $b_D \cdot b = z$. And by (2) we have $\min \| b_D - b \| \leqq n^{1/p_B} \cdot s_{max} = n^{1/p_B} \cdot o(\| z/n \|^{1/p_B}) = o(\| z \|^{1/p_B})$ as $n \to \infty$. Thus $\min \| b^2 - z \| = \min \| b \cdot b - b_D \cdot b \| \to 0$ as $n \to \infty$. The theorem is proved.

Of course Theorem 5.1.1 is interesting only when $p_B > 1$. Now we shall say that a map T from a metric space into a metric space satisfies a first order Lipschitz condition for large distances if for every $\varepsilon > 0$ there is a K such that $d(T(x_1), T(x_2)) \leqq K \cdot d(x_1, x_2)$ if $d(x_1, x_2) \geqq \varepsilon$. It is easy to see that a uniformly continuous map from a convex set in a Banach space into a metric space satisfies a first order Lipschitz condition for large distances. (3)

In the following theorem we weaken the condition on the group operation to uniform continuity only. The theorem will be of use to derive the results of Chapter 6.

THEOREM 5.1.2. *If in a commutative Banach group* B, (1) $p_B > 1$, (2) $(x, y) \to xy$ *is uniformly continuous, then we can introduce a group invariant metric* d_ε *in* B *such that* (a) $d_\varepsilon(x, y) \geqq \| x - y \|$ *for all* x *and* y *in* B *and* d_ε *gives the same uniform structure on* B *as the norm*: (b) *there exist two constants* K_1 *and* K_2 *such that for every* $z \in B$ *there exists a* $b \in B$ *such that* $d_\varepsilon(b^2, z) \leqq K_1$ *and* $|2 \cdot d_\varepsilon(b, 0) - d_\varepsilon(b^2, 0)| \leqq K_2 (d_\varepsilon(z, 0))^{1/p_B}$.

PROOF. We put $d'(x, y) = \sup \| xz - yz \|$. Since the group multiplication is uniformly continuous, d' is uniformly equivalent to the norm and $d'(x, y) \geqq \| x - y \|$. And d' is a group invariant metric. Now choose an $\varepsilon > 0$ and let $d_\varepsilon(x, y)$ be the infimum of the lengths of the ε-chains between x and y in the metric d'. Then d_ε is group invariant and d_ε gives the same uniform structure on B as d'. For $d_\varepsilon(x, y) = d'(x, y)$ if $d'(x, y) \leqq \varepsilon$. We also have $d_\varepsilon(x, y) \geqq d'(x, y) \geqq \| x - y \|$.

And by (3) there is a K such that $\| x - y \| \leqq d_\varepsilon(x,y) \leqq K \cdot \| x - y \|$ if $d_\varepsilon(x,y) \geqq \varepsilon$. Thus (a) is proved. Choose K such that $d_\varepsilon(x,y) \leqq K\varepsilon$ if $\| x - y \| \leqq \varepsilon$.

Let z be a point in B and let $0 = x_0, x_1, \cdots, x_n = z$ be an ε-chain whose length is $< d_\varepsilon(z,0) + \varepsilon$. We form $y_m = x_m \cdot x_{m-1}^{-1}$, then $y_1 y_2 \cdots y_n = z$. We can assume that there is at most one j such that $d_\varepsilon(y_j, e) < \varepsilon/2$, otherwise by multiplying together some of the y_ν : s we can obtain that this is fulfilled. We thus have $n\varepsilon \geqq d_\varepsilon(z,0) \geqq (n/2 - 2)\varepsilon$. In the same way as in the proof of Theorem 5.1.1 we now form the n-dimensional cube with the 2^n points $0 \cdot y_{k_1} y_{k_2} \cdots y_{k_j}$. In the norm metric we have $s_{max} \leqq \varepsilon$ in this cube. Thus we have $\min \| b_{1D} - b_1 \| \leqq n^{1/p_B} \cdot \varepsilon$ in this cube and so $\min d_\varepsilon(b_{1D}, b_1) \leqq K \cdot n^{1/p_B} \cdot \varepsilon$. Since d_ε is invariant and $d_\varepsilon(z,0) \leqq \sum d_\varepsilon(y_m, 0) \leqq d_\varepsilon(z,0) + \varepsilon$ we have $d_\varepsilon(z,0) \leqq d_\varepsilon(b_{1D}, 0) + d_\varepsilon(b_1, 0) \leqq d_\varepsilon(z,0) + \varepsilon$. Since $d_\varepsilon(b_1,0) - K \cdot n^{1/p_B} \cdot \varepsilon \leqq d_\varepsilon(b_{1D}, 0) \leqq d_\varepsilon(b_1, 0) + K \cdot n^{1/p_B} \cdot \varepsilon$ we have

$$d_\varepsilon(z,0) - K \cdot n^{1/p_B} \cdot \varepsilon \leqq 2 \cdot d_\varepsilon(b_1, 0) \leqq d_\varepsilon(z,0) + (K \cdot n^{1/p_B} + 1)\varepsilon. \qquad (4)$$

$$d(b_1^2, z) = d_\varepsilon(b_{1D}, b_1) \leqq K \cdot n^{1/p_B} \cdot \varepsilon. \qquad (5)$$

Now we repeat the process above with $z_1 = z \cdot b_1^{-2}$ instead of z. Then we get $n_1\varepsilon \geqq d_\varepsilon(z_1, 0) \geqq ((n_1/2) - 2)\varepsilon$. And we get an element b_2 which satisfies $d(b_2^2, z_1) \leqq K \cdot n_1^{1/p_B} \cdot \varepsilon$, $2 \cdot d_\varepsilon(b_2, 0) \leqq d\varepsilon(z_1, 0) + (K \cdot n_1^{1/p_B} + 1)\varepsilon$. And then we repeat the process with $z_2 = z \cdot b_1^{-2} \cdot b_2^{-2}$. and so on. We finsh the constructions when we get to the first number m such that $(n_m/2 - 2)\varepsilon < 2 \cdot (K \cdot n_m^{1/p_B} + 1)\varepsilon$ which certainly will occur. Since the reverse inequality holds for all sufficiently large n_m: s there is a constant K_1 such that $K_1 \geqq n_m\varepsilon \geqq d_\varepsilon(z_m, 0) = d_\varepsilon(z \cdot b_1^{-2} \cdot \cdots \cdot b_m^{-2}, 0) = d_\varepsilon(z, b_1^2 \cdot b_2^2 \cdot \cdots \cdot b_m^2)$. Since by (4) we have $d_\varepsilon(b_j, 0) \leqq d_\varepsilon(z_{j-1}, 0)$, $2 \leqq j \leqq m$, we have $d_\varepsilon(b_2 b_3 \cdots b_m, 0) \leqq \sum_{j=2}^m d_\varepsilon(b_j, 0) \leqq \sum_{j=1}^{m-1} d_\varepsilon(z_j, 0) \leqq (\sum_0^{m-2} 2^{-j}) \cdot d_\varepsilon(z_1, 0) \leqq 2 \cdot d_\varepsilon(z_1, 0)$. This last inequality and (4) give that there is a constant K_2 such that $|2 \cdot d_\varepsilon(b_1 b_2 \cdots b_m, 0) - d_\varepsilon(b_1^2 b_2^2 \cdots b_m^2, 0)| \leqq K_2 \cdot (d_\varepsilon(z,0))^{1/p_B}$ and thus $b = b_1 b_2 \cdots b_m$ satisfies the conditions of the theorem.

Theorem 5.1.2 gives that the group resembles a Banach space for large distances. We get the

COROLLARY. *If a subgroup G of a metric linear space is uniformly homeomorphic to a Banach space with roundness > 1, then in G $\sup_z d(z, M) < \infty$ where M is the set of elements of the form $2y$.*

PROOF. Let $T : G \to B$ be a uniform homeomorphism. Then B with the group structure inducted by T becomes a Banach group which satisfies the conditions of Theorem 5.1.2. Thus in B $\sup_y d(y, T(M)) < \infty$ in the norm metric and since

T^{-1} with respect to this metric satisfies a first order Lipschitz condition for large distances the corollary is proved.

EXAMPLE 5.1.1. The following closed connected subgroup of $L_2(0,1)$ is not uniformly homeomorphic to any Banach space with roundness > 1 (as a consequence of the corollary above). Let the group be the closed hull of the following set $G: G$ consists of the $L_2(0,1)$-functions whose values on the interval $(1/(n+1), 1/n)$ are multiples of n^2 for every $n \geqq 1$.

If G is a subgroup of the additive group of some metric linear space and G is not a linear space, can G then be uniformly homeomorphic to a Banach space with roundness > 1? This question is of central importance in the further development of the theory. In view of Theorem 5.1.2 it seems natural to study subgroups of Banach spaces, i.e. the Example 5.2.1 of this paper.

We now give the counterpart of Theorem 5.1.2 for uniformly convex Banach spaces.

THEOREM 5.1.3. *If in a commutative Banach group B,* (1) *B is uniformly convex,* (2) *$(x,y) \to xy$ is uniformly continuous then we can introduce a group invariant metric d_ε in B such that,* (a) *$d_\varepsilon(x,y) \geqq \| x - y \|$ for all x and y in B and d_ε gives the same uniform structure on B as the the norm,* (b) *for every $\delta > 0$ there is an w such that for every z with $d_\varepsilon(z,0) > w$ there is a $b \in B$ such that $d_\varepsilon(b^2, z) \leqq \delta \cdot (d_\varepsilon(z,0))$ and $|2 \cdot d_\varepsilon(b,0) - d_\varepsilon(b^2,0)| \leqq \delta \cdot (d_\varepsilon(z,0))$.*

PROOF. We define d_ε in exactly the same way as in the proof of Theorem 5.1.2. This gives (a). We then consider a $z \in B$ and let $0 = x_0, x_1 \cdots, x_n = z$ be an ε-chain between 0 and z with length $< d_\varepsilon(z,0) + \varepsilon$. We form $y_m = x_m \cdot x_{m-1}^{-1}$ and as in the proof of Theorem 5.1.2 we can assume that there is at most one m with $d_\varepsilon(y_m, 0) < \varepsilon/2$. As in the proof of Theorem 5.1.2 we now form the n-dimensional cube with the 2^n points $0 \cdot y_{k_1} y_{k_2} \cdots y_{k_j}$. Now if K_1 is an arbitrary real number and n is sufficiently large (how large depends on K_1) then there is by Theorem 4.1.6 an m_1-diagonal, $m_1 \geqq m$, in this cube with length $\leqq m_1 \cdot \varepsilon/K_1$ in the norm metric and thus with length $m_1 \cdot \varepsilon \cdot K/K_1$ in the metric d_ε where K is the constant which appears in the proof of Theorem 5.1.2. We assume that the points of this m_1-diagonal can be written $y_{11} y_{12} \cdots y_{1n_1} \cdot x$ and $y_{21} y_{22} \cdots y_{2n_2} \cdot x$ where $n_1 + n_2 = m_1$. Then $d_\varepsilon(y_{11} y_{12} \cdots y_{1n_1}, y_{21} y_{22} \cdots y_{2n_2}) \leqq m_1 \cdot \varepsilon \cdot K/K_1$. We put $y_{11} y_{12} \cdots y_{1n_1} = b_{11}$ and $y_{21} y_{22} \cdots y_{2n_2} = b_2$. We consider $z_1 = z \cdot b_{11}^{-1} \cdot b_{12}^{-1}$. We can assume $z_1 = y_1 y_2 \cdots y_{(n-m_1)}$ otherwise we can renumber the $y_\nu : s$ so that this is the case. And we form the $(n - m_1)$-dimensional cube which consists of the 2^{n-m_1} elements $0 \cdot y_{k_1} y_{k_2} \cdots y_{k_j}$ where $k_j \leqq n - m_{1 1}$. Now if $d_\varepsilon(z_1, 0)$ is sufficiently large also in the $(n - m_1)$-dimensional cube described above we can find an m_2-diagonal, $m_2 \geqq m$, whose length in d_ε is $\leqq m_1 \cdot \varepsilon \cdot K/K_1$. We form b_{21} and b_{22} in the same way as we formed b_{11} and b_{12}. We then consider $z_2 = z_1 \cdot b_{21}^{-1} \cdot b_{22}^{-1}$ and if $d(z_2, 0)$ is sufficiently large we can repeat the process. We repeat the process so many times that there is no more m_j-diagonal, $m_j \geqq m$, with length $\leqq m_j \cdot \varepsilon \cdot K/K_1$ in d_ε in the cube we get. For every $\delta_1 > 0$ there is an w_1 such that if $d_\varepsilon(z, 0) > w_1$ we have at this stage $z = b_{11} b_{21} \cdots b_{j1} \cdot b_{12} b_{22} \cdots b_{j2} \cdot z_j$ where $d_\varepsilon(z_j, 0) \leqq \delta_1 \cdot d_\varepsilon(z, 0)$. And we have $d_\varepsilon(b_{11} b_{21}\, b_{j1}, b_{12} b_{22}\, b_{j2}) \leqq d_\varepsilon(z, 0) \cdot (3 \cdot \varepsilon \cdot K/K_1)$. By putting $b = b_{11} b_{21} \cdots b_{j1}$ we see that the theorem is proved.

If $d_\varepsilon(z, b^2)$ is sufficiently large we can make the same construction of a square root of the element $y = z \cdot b^{-2}$ as we made when we constructed b from z an by repeating this approximation process of square roots we obtain

THEOREM 5.1.4. *If in a commutative Banach group B,* (1) *B is uniformly convex,* (2) *$(x, y) \to xy$ is uniformly continuous then* $\sup_x d(x, M) < \infty$ *where M is the set of element of the form y^2.*

We also get the following corollary which we give a slightly more general formulation than the corollary of Theorem 5.1.2.

COROLLARY. *If a commutative metric group is uniformly homeomorphic to a uniformly convex Banach space then in the group* $\sup_z d(z, M) < \infty$ *where M is the set of elements of the form y^2.*

The methods of constructing square roots described above give quite exact information on the existence and largeness of square roots when we strengthen the condition on the group multiplication to satisfy locally a first order Lipschitz condition. This is shown by the theorem below.

THEOREM 5.1.5. *If in a commutative local Banach group U* (1) *the Banach space is uniformly convex or has roundness* > 1, (2) $\| x_2 y - x_1 y \| \leqq K \| x_2 - x_1 \|$ *for all x_1, x_2 and y in some neighbourhood of* 0 *for some K, then we can introduce a group invariant metric d_1 in some neighbourhood of* 0 *such that* (a) $K \| x - y \| \geqq d_1(x, y) \geqq \| x - y \|$ *for all x and y in some neighbourhood of* 0 (b) *to every z in some neighbourhood of* 0 *and every $\varepsilon > 0$ we can find a b such that $b^2 = z$ and* $|d_1(z, 0) - 2 \cdot d_1(b, 0)| < \varepsilon$.

PROOF. We carry out the proof in detail only for the case when $p_B > 1$. If U_1 is a sufficiently small neighbourhood of 0 we can define $d'(x,y) = \sup \| xz - yz \|$ which gives an invariant écart in U_1. Then we can define $d_1(x,y)$ to be the infimum of the lengths in d' of the arcs between x and y in U_1. Then d_1 is defined and gives an invariant metric in some neighbourhood U_2 of 0. Then d_1 satisfies (a) which we see by considering the arc between x and y which is the linear segment in the Banach space which connects x and y.

Now let z be an element sufficiently near 0, put $d_1(z,0) = r$. If δ is a positive real number we can consider an arc between 0 and z with length $< r + \delta$ and for every integer $n \geqq 1$ we can consider a sequence of points $0 = x_0, x_1, x_n = z$ on the arc such that $d_1(x_i, x_{i+1}) < (r + \delta)/n$ for all i. Put $y_m = x_m \cdot x_{m-1}^{-1}$. For every n we consider the n-dimensional cube with the 2^n points $0 \cdot y_{k_1} y_{k_2} \cdots y_{k_j}$. In this cube the largest edge has length $< (r + \delta)/n$ in the norm metric and thus the smallest n-diagonal (b_{1D}, b_1) has length $< ((r + \delta) \cdot K \cdot n^{1/p})/n$ in d_1 and this tends to 0 as $n \to \infty$. However, to get a b with $b^2 = z$ we have to repeat the process above with $z \cdot b_1^{-2}$. Then we can get a convergent series $b_1, b_1b_2, b_1b_2b_3, \cdots$ which converges to a b with $b^2 = z$. And if $\varepsilon > 0$ is given we can choose the length of the first arc to be so near r and let the convergence to b be so fast that $|d_1(z,0) - 2 \cdot d_1(b,0)| < \varepsilon$. The proof in the case of a uniformly convex Banach space goes in the same way with obvious modifications.

We shall say that two metric spaces are Lipschitz-equivalent if there is a one-to-one mapping $T: B_1$ onto B_2 such that T and T^{-1} satisfy a first order Lipschitz condition. We shall say that two metric groups are locally Lipschitz-equivalent if some neighbourhood of e in one of the groups is Lipschitz-equivalent to some neighbourhood of e in the other group. Theorem 5.1.5 gives the

COROLLARY. *If a locally generated subgroup G of some metric linear space is locally Lipschitz-equivalent to a Banach space, which is uniformly convex or has roundness > 1, then G is a normable topological linear space.*

We think that very little can be said about the existence of square roots in commutative Banach groups without some geometrical condition on the Banach space. However, we give the following

THEOREM 5.1.6. *If in a commutative Banach group , $\| x_2y - x_1y \| \leqq K\| x_2 - x_1 \|$ where $K < 2$, then every element of the group has a square root.*

PROOF. Put $d'(x,x_0) = \sup_y \| yx - yx_0 \|$. Then d' is a group invariant metric in B and $K \cdot \| x - x_0 \| \geqq d'(x, x_0) \geqq \| x - x_0 \|$. Now for $z \in B$ we construct a sequence y_n which converges to a square root of z in the following way: put $y_0 = 0$ and if y_n is defined put $y_{n+1} = (y_n + z \cdot y_n^{-1})/2$. Then we have $\| y_n - zy_{n+1}^{-1} \| \leqq d'(y_n, zy_{n+1}^{-1}) = d'(y_{n+1}, zy_n^{-1}) = d'((y_n + z \cdot y_n^{-1})/2, zy_n^{-1})$

$$\leqq K \cdot \| (y_n + zy_n^{-1})/2 - zy_n^{-1} \| = K/2 \| y_n - zy_n^{-1} \| \tag{a}$$

Also $\| zy_n^{-1} - zy_{n+1}^{-1} \| \leqq d'(zy_n^{-1}, zy_{n+1}^{-1}) = d'(y_n, y_{n+1}) \leqq K \cdot \| y_n - y_{n+1} \|$

$$= K/2 \| y_n - zy_n^{-1} \| \tag{b}$$

(a) and (b) give $\| (y_n + zy_n^{-1})/2 - zy_{n+1}^{-1} \| \leqq K/2 \| y_n - zy_n^{-1} \|$ that is $\| y_{n+1} - zy_{n+1}^{-1} \| \leqq K/2 \| y_n - zy_n^{-1} \|$. This inequality immediately gives $y_n^2 \to z$ as $n \to \infty$. This inequality also gives that y_n is a convergent sequence for $\| y_{n+2} - y_{n+1} \| = 1/2 \| y_{n+1} - zy_{n+1}^{-1} \| \leqq K/2 \cdot 1/2 \| y_n - zy_n^{-1} \| = K/2 \| y_{n+1} - y_n \|$.

We think that Theorem 5.1.6 becomes false if we put $K = 2$, but we have not succeeded in constructing a counter-example. The problem of constructing a counter-example is a part of Problem 6.1.1 of the next chapter.

5.2. Groups generated by arcs in Banach spaces. We shall say that a continuous arc $t \to x(t)$, $x(0) = 0$, $0 \leqq t \leqq 1$ in a Banach space B generates G if G is the smallest closed subgroup of B which contains $\{x(t)\}$. It is well-known that every continuous arc $t \to x(t)$, $x(0) = 0$, $0 \leqq t \leqq 1$, in a finite-dimensional Banach space generates a linear subspace of the Banach space. The theorems and example below give information about the infinite-dimensional case.

THEOREM 5.2.1. *If for the arc $t \to x(t)$, $x(0) = 0$, $0 \leqq t \leqq 1$ $\| x(t_2) - x(t_1) \| = o(\| t_2 - t_1 \|^{1/p_B})$ holds as $| t_2 - t_1 | \to 0$ then $t \to x(t)$ generates a linear subspace of B.*

PROOF. It is sufficient to prove that in the group generated by $t \to x(t)$ there are elements arbitrarily close to $(x(t))/2$ for every t in $[0,1]$. Consider an $x(t_0)$ and an n and form the elements $y_m = x(mt_0/n) - x((m-1)t_0/n)$, $1 \leqq m \leqq n$. Then $y_1 + y_2 + \cdots + y_n = x(t_0)$. We form the n-dimensional cube formed by the 2^n elements $0 + y_{k_1} + y_{k_2} + \cdots + y_{k_j}$ where in every sum y_ν appears once or not at all. In this cube we have by assumption $s_{max} = o((1/n)^{1/p_B})$ as $n \to \infty$ and thus $d_{min} = \min \| b_D - b \| \leqq s_{max} \cdot n^{1/p_B} = o(1)$ as $n \to \infty$. Since $b_D + b = x(t_0)$ this gives that there are elements arbitrarily close to $x(t_0)/2$ in the group generated by $t \to x(t)$. The theorem is proved.

EXAMPLE 5.2.1. For $1 \leqq p \leqq 2$ consider the arc in $L_p(0,1)$ where $x(t)$ is the function $f(y)$ which is 1 on the interval $0 \leqq y \leqq t$ and 0 on the interval $t < y \leqq 1$.

Then $\| x(t_2) - x(t_1) \| = | t_2 - t_1 |^{1/p}$ and $t \to x(t)$ does not generate a linear subspace of $L_p(0,1)$. Since $L_p(0,1)$, $1 \leqq p \leqq 2$ has roundness p, the example shows that theorem 5.2.1 is in a sense the best possible.

With an obvious modification of the proof we get

THEOREM 5.2.1. *If for an arc* $t \to x(t), x(0) = 0$, $0 \leqq t \leqq 1$, *in a uniformly convex Banach space* B, $\| x(t_2) - x(t_1) \| \leqq K \cdot | t_2 - t_1 |$ *for all* $t_1, t_2 \in [0,1]$ *and some* K, *then* $t \to x(t)$ *generates a linear subspace of* B.

Theorem 5.2.1 suggests the problem if the following smallness property is characteristic for Banach spaces isomorphic to a Hilbert space: every arc $t \to x(t)$, $x(0) = 0$, $0 \leqq t \leqq 1$, in the Banach space which satisfies $\| x(t_2) - x(t_1) \| = o(| t_2 - t_1 |^{1/2})$ as $| t_2 - t_1 | \to 0$ generates a linear subspace of the Banach space.

6. Uniform homeomorphisms and isomorphisms between topological linear spaces

6.1. In this chapter we turn from the more general metric commutative groups to metric linear spaces. One of the results of the chapter shows that a locally bounded linear space which is uniformly homeomorphic to a Hilbert space is isomorphic to the Hilbert space. There are several reasons for considering questions of this type. First, in the corollary of Theorem 5.1.5 and in the theorems of Chapter 7 of this paper we will arrive at a situation where we have two metric linear spaces which are locally Lipschitz-equivalent or uniformly homeomorphic and so the question inevitably arises if they are isomorphic. Secondly, it is well-known that two finite-dimensional topological linear spaces which are locally homeomorphic are isomorphic and so it is natural to ask if there is some corresponding theorem for infinite-dimensional topological linear spaces. It has been proved by Kadec [14] that all separable Banach spaces are homeomorphic and in Mazur [15] it is proved that the $L_p(0,1)$-spaces, $1 \leqq p < \infty$, are locally uniformly homeomorphic and thus none of these conditions will imply linear isomorphism. In Lindenstrauss [7], Henkin [16] and Enflo [2] and [3] several examples of Banach spaces are given which are not uniformly homeomorphic and so uniform equivalence seems to be a natural condition. The problems just mentioned and also some other problems in the paper may all be regarded as special cases of the following general

PROBLEM 6.1.1. To what extent is the following true: two connected uniform groups with the same underlying uniform space are isomorphic.

Problem 6.1.1 can also of course be investigated with some Lipschitz condition or in the case of non-commutative groups with some extra differentiability or analyticity condition. As a partial problem one can try to determine if a uniform group which is Lipschitz-equivalent to some appropriate Banach space (in some two-sided invariant metric) is commutative.

6.2. Locally bounded linear spaces and Banach spaces.

THEOREM 6.2.1. *If a locally bounded linear space is uniformly homeomorphic to a Banach space with roundness* > 1, *then it is a normable space*

PROOF. We begin with two lemmas.

LEMMA 1. *If* d' *is an invariant metric in a locally bounded linear space* M, *then there is a* $\delta > 0$ *such that, if a new metric* d_ε *is introduced in* M *by letting* $d_\varepsilon(x, y)$ *be the infimum of the lengths of the* ε*-chains between* x *and* y, $\delta \geqq \varepsilon$, *then there are constants* C *and* w *such that* $d_\varepsilon(2b, 0) \geqq C \cdot d_\varepsilon(b, 0)$ *if* $d_\varepsilon(b, 0) \geqq w$.

PROOF. Let U be a bounded, balanced neighbourhood of 0 in M, choose δ such that $d'(x, 0) \leqq \delta \Rightarrow x \in U$ and choose $\varepsilon = \delta$. Then there is a K such that $x \in U \Rightarrow d_\varepsilon(x, 0) \leqq K\varepsilon$. Now let $z \in M$ and let $x_0, x_1, \cdots, x_n$ be an ε-chain in d' between 0 and z, with length $< d_\varepsilon(z, 0) + \varepsilon$ and such that there is at most one i with $d'(x_i, x_{i+1}) < \varepsilon/2$. Then $(n/2 - 2)\varepsilon \leqq d_\varepsilon(z, 0) \leqq n\varepsilon$. Now since U is balanced $x_0, x_1/2, x_2/2, \cdots x_n/2$ is a $K\varepsilon$-chain in d_ε between 0 and $z/2$ and thus $d_\varepsilon(z/2, 0) \leqq Kn\varepsilon$. This proves the lemma.

LEMMA 2. *Let* d' *be an invariant metric in a locally bounded linear space and let* d_ε *be defined as in Lemma* 1 *where* ε *is chosen such that* $d'(x, 0) \leqq \varepsilon \Rightarrow x \in U$ *for some bounded, balanced neighbourhood* U *of* 0. *Then for every* $\delta > 0$ *and* $w > 0$ *there is a real* N *such that* $d_\varepsilon(x, 0) > \delta \Rightarrow d(\alpha x, 0) > w$ *if* $\alpha > N$.

PROOF. Since $U_1 = \{x \,|\, d'(x, 0) \leqq \varepsilon\}$ is bounded the family of sets $1/t \cdot U_1$, $t > 0$ forms a fundamental system of neighbourhoods of 0. Thus $1/t \cdot U_1 + 1/t \cdot U_1 \subset U_1$ for some $t > 0$ and so $U_1 + U_1 \subset tU_1$. The last inclusion gives $U_1 + U_1 + U_1 + U_1 \subset tU_1 + tU_1 \subset t^2 U_1$ and so by induction we have $U_1 + U_1 + \cdots + U_1 (2^n U_1 : s) \subset t^n \cdot U_1$. On the left side in the last inclusion we have all points of M at distance $\leqq 2^n \cdot \varepsilon$ from 0. Now if for some $m > 0$ we have $x \notin mU$ then $\alpha x \notin t^n \cdot U_1$ if $\alpha > t^n/m$ and so $d_\varepsilon(\alpha x, 0) > 2^n \cdot \varepsilon$ in this case. Since the family of sets $m \cdot U, m > 0$, forms a fundamental system of neighbourhoods of 0 the lemma is proved.

PROOF OF THEOREM 6.2.1. Let T be a uniform homeomorphism from the locally bounded linear space M onto the Banach space B. Then B with the group structure induced by T satisfies the conditions of Theorem 5.1.2. Thus we introduce first an invariant metric d' in M and then we introduce an invariant metric d_ε in M by letting $d_\varepsilon(x,y)$ be the infimum of the lengths of the ε-chains between When in the Proof of 5.1.2 we define $d_\varepsilon(x,y)$ we can choose ε to be an arbitrary positive number and now we choose it so small that the ε-sphere around 0 in d' is contained in a bounded, balanced neighbourhood of 0. Now Theorem 5.1.2 gives that there are numbers K_1 and K_2 such that for every $z \in M$ there is an element of the form $2b$ at d_ε-distance $\leqq K_1$ from z and with $|2 \cdot d_\varepsilon(b,0) - d_\varepsilon(2\,b,0)| \leqq K_2 \cdot (d_\varepsilon(z,0))^{1/p_B}$. The continuity property of $x \to x/2$ proved in Lemma 1 now gives that there exists a constant K_3 such that $|2 \cdot d_\varepsilon(z/2,0) - d_\varepsilon(z,0)| \leqq K_3 \cdot (d_\varepsilon(z,0))^{1/p_B}$ for every $z \in M$. We define $d_n(x,0) = \overline{\lim}_{t\to\infty}(d_\varepsilon(tx_1,0))/t$. Then $d_n(x+y,0) \leqq d_n(x,0) + d_n(y,0)$ and $d_n(\alpha x,0) = |\alpha| \cdot d_n(x,0)$ for real $\alpha : s$. Thus d_n defines a seminorm on M. We have obviously $d_n(x,0) \leqq d_\varepsilon(x,0)$ and Lemma 2 and the inequality $|2 \cdot d(z/2,0) - d_\varepsilon(z,0)| \leqq K_3 \cdot (d_\varepsilon(z,0))^{1/p_B}$ give that d_n defines a norm on M which gives the same topolgy as d_ε. The theorem is proved.

As a consequence of Theorem 6.2.1 we see that $L_p(0,1)$ is not uniformly homeomorphic to $L_q(0,1)$ if $0 < p < 1$, $1 < q < \infty$.

THEOREM 6.2.2. *If a locally bounded linear space is uniformly homeomorphic to a uniformly convex Banach space, then an invariant metric d_ε can be introduced in it such that $\lim_{n\to\infty}(d_\varepsilon(2nx,0))/(d_\varepsilon(nx,0)) = 2$, for all x and the convergence is uniform in the set set $U_\delta = \{x \mid d_\varepsilon(x,0) > \delta\}$ for every δ.*

The proof of this goes in the same way as the proof of Theorem 6.2.1 but since Theorem 5.1.3 is weaker than Theorem 5.1.2 we cannot successfully introduce the metric d_n in this case.

6.3. Spaces uniformly homeomorphic to a Hilbert space

THEOREM 6.3.1. *If a Banach space is uniformly homeomorphic to a Hilbert space, then it is isomorphic to the Hilbert space.*

PROOF. The theorem follows from the lemmas below. If T is a Lipschitz map between teo metric spaces then we put $\|T\| = \sup(d(T(x),T(y)))/(d(x,y))$. If X and Y are metric spaces we shall say that X is Lipschitz embeddable in Y if X is Lipschitz-equivalent to some subset of Y. We say that the map T which gives the Lipschitz-equivalence between X and a subset of Y is a Lipschitz embedding of X in Y. Our first lemma is a generalisation of a well-known theorem on isometric embeddings in Hilbert space.

LEMMA 1. *If X is a separable metric space, then there exists a Lipschitz embedding T of X in Hilbert space with $\| T \| \, \| T^{-1} \| \leqq K$ if and only if for every finite subset M of X there is a Lipschitz embedding T_M of M in Hilbert space such that $\| T_M \| \, \| T_M^{-1} \| \leqq K$.*

PROOF. Let $\{a_\gamma\}$ be a sequence such that $\{a_\gamma\}$ is dense in X and let the Hilbert space be represented as 1_2 (i.e. the space of real square-summable number sequences). Now suppose that every finite subset M of X is Lipschitz embeddable in Hilbert space such thaat $\| T_M \| \, \| T_M^{-1} \| \leqq K$. Let M_n be $\{a_\nu \,|\, , 1 \leqq \gamma \leqq n\}$ and le^t E_n be the n-dimensional subspace of 1_2 which has the property that all numbers after the $n : th$ in all sequences of E_n are 0. For all M_n and all a_γ we can choose T_{M_n} in such a way that $T_{M_n}(a_\gamma) \in E_\gamma$, such that $T_{M_n}(a_1) = 0$ for all n and such that $\| T_{M_n} \| = 1$ for all n. We can find a sequence of positive integers n_ν such that $T_{M_n}(a_2)$ converges for this sequence and then we can find a subsequence of $\{n_\gamma\}$ such that $T_{M_n}(a_3)$ converges for this subsequence. By repeating this process for every a_j and then finally choose a diagonal subsequence we obtain that for this diagonal subsequence $T_{M_n}(a_j)$ converges for all a_j. The limit gives an embedding T of $\{a_\gamma\}$ with the required properties and since $\{a_\gamma\}$ is dense in X it can be extended to an embedding of all X in Hilbert space such that $\| T \| \, \| T^{-1} \| \leqq K$. The lemma is proved.

LEMMA 2. *If a Banach space B is uniformly homeomorphic to a Hilbert space then there is a K such that for every finite-dimensional subspace C of B there is a Lipschitz embedding T of C in Hilbert space with $\| T \| \, \| T^{-1} \| \leqq K$.*

PROOF. Let T be a uniform homeomorphism from B onto the Hilbert space. Then T and T^{-1} both satisfy a Lipschitz condition for large distances say with the Lipschitz constants K_1 and K_2. Let M be a finite subset of C. Then by multiplying all vectors in M by a number n we get a set nM. If n is sufficiently large then there is a Lipschitz embedding T_{nM} of nM in Hilbert space such that $\| T_{nM} \| \, \| T_{nM}^{-1} \| \leqq K_1 \cdot K_2$, namely the Lipschitz embedding defined by the uniform homeororphism. Thus there is a Lipschitz embedding T_M of M in Hilbert space such that $\| T_M \| \, \| T_M^{-1} \| \leqq K_1 \cdot K_2$. Now Lemma 1 applies to C and so Lemma 2 is proved.

LEMMA 3. *If $t \to x(t)$ from $[0, 1]$ into Hilbert space satisfies a first order Lipschitz condition then it has a derivative for almost all t.*

PROOF. We can assume that the Hilbert space is separable. We represent it as 1_2 and consider $t \to (x_1(t), x_2(t), \cdots)$. Then for almost all t, $x_j'(t)$ exists for every j. This holds since for every j, $t \to x_j(t)$ satisfies a first order Lipschitz condition. If for some t_0, $x_j'(t_0)$ exists for all j, then $(x_1'(t_0), x_2'(t_0), \cdots)$ is an element of 1_2, otherwise $t \to x(t)$ would not satisfy a first order Lipschitz condition. However, $(x_1'(t_0), x_2'(t_0), \cdots)$ is the derivative of $t \to x(t)$ at t_0 if and only if as $t \to t_0$ and $j \to \infty$ we have $\| (0,0,\cdots,0,x_j(t),x_{j+1}(t),\cdots) - (0,0, \ ,0,x_j(t_0),x_{j+1}(t_0)\cdots) \| = o(|t - t_0|)$. The set of $t : s$ where this does not hold is obviously measurable and we assume that it has positive measure. Then, since for every $\delta > 0$, $t \to x_j(t)$ is continuous in a set of measure $> 1 - \delta$ we have a set M of $t : s$ with the following properties: (a) M has positive measure, (b) for every j, $x_j'(t)$ exists for all t in M and $x_j'(t)$ is continuous in M, (c) there is an $\varepsilon > 0$ such that for all t_0 in M and all $\gamma > 0$ and $w > 0$ there are t and j, $|t - t_0| < \gamma$, $j > w$ such that

$$\| (0,0,\cdots,0,x_j(t),x_{j+1}(t),\cdots) - (0,0,\cdots,0,x_j(t_0),x_{j+1}(t_0),\cdots) \| > \varepsilon |t - t_0|.$$

Now choose a t_0, $t_0 \in M$, such that the average density of M at t_0 is 1. We consider an interval to the right of t_0, an integer j_1, and a t_1 in this interval such that for some N_1

$$\| (0,0,\cdots,0,x_{j_1}(t_1),x_{j_1+1}(t_1),\cdots,x_{N_1}(t_1),0,0,\cdots) - (0,0,\cdots,0,x_{j_1}(t_0),x_{j_1+1}(t_0),\cdots,x_{N_1}(t_0),0,0,\cdots) \| > \varepsilon |t_1 - t_0|$$

Here we can first choose j_1 arbitrarily large and then t_1 arbitrarily near t_0. Now since the average density of M at t_0 is 1 and since

$$(0,0,\cdots,0,x_{j_1}'(t),x_{j_1+1}'(t),\cdots,x_{N_1}'(t),0,0,\cdots)$$

is continuous in M, if $|t_1 - t_0|$ is sufficiently small then we must have

$$\| (0,0,\cdots,0,x_{j+1}'(t),x_{j_1+1}'(t),\cdots,x_{N_1}'(t),0,0,\cdots) \| \geqq \varepsilon$$

in a set M_1 of positive measure, $M_1 \subset (M \cdots [t_0, t_1])$. Now we can repeat the process above with M_1 instead of M. Since we could choose j_1 arbitrarily large we choose this time $j_2 > N_1$ and so we find a set M_2 of positive measure, $M_2 \subset M_1$, such that

$$\| (0,0, \ ,0,x_{j_1}'(t),x_{j_1+1}'(t), \ ,x_{N_1}'(t),0,0, \ ,0,x_{j_2}'(t),x_{j_2+1}'(t), \ ,x_{N_2}'(t),0,\cdots \| \geqq \varepsilon\sqrt{2}$$

in M_2. By repeating the process sufficiently large number of times we get a contradiction to the fact that $t \to x(t)$ satisfies a first order Lipschitz condition. The lemma is proved.

In the next two lemmas we will consider finite-dimensional Banach spaces.

LEMMA 4. *If, for an n-dimensional Banach space B, there is a Lipschitz embedding* T_m, $0 \leqq m < n$, *of B into Hilbert space such that (a)* T_m *is linear on some m-dimensional subspace* C_m *of B and* T_m *is linear on every* $(m+1)$*-dimensional subspace of B which contains* C_m

(b) $$\| T_m \| \; \| T_m^{-1} \| \leqq K$$

then there is a Lipschitz-embedding of B into Hilbert space such that (a$_1$) T_{m+1} *is linear on some* $(m+1)$*-dimensional subspace* C_{m+1} *of B and* T_{m+1} *is linear on every* $(m+2)$*-dimensional subspace of B which contains* C_{m+1}

(b$_1$) $$\| T_{m+1} \| \; \| T_{m+1}^{-1} \| \leqq K$$

PROOF. Let N be a countable set of points on the boundary of the unit sphere of B such that N is dense on the boundary of the unit sphere. It follows from Lemma 3 that for almost all x in B, T_m has a derivative in all directions of vectors of N at x. Let x be such a point, $x \notin C_m$. If T_m has a derivative $T'_{m,a}$ in the direction on the vector a at x, we put $T_{m+1}(x + va) = T_m(x) + v \cdot T'_{m,a}$ for real v : s. This defines T_{m+1} on a dense subset of B. We have on this dense subset

$$T_{m+1}(x + v_1 a_1) - T_{m+1}(x + v_2 a_2) = \lim_{r \to 0} \frac{T_m(x + r v_1 a_1) - T_m(x + r v_2 a_2)}{r}$$

This equation gives that T_{m+1} can be extended by continuity to all of B such that $\| T_{m+1} \| \; \| T_{m+1} \| \leqq K$. We assume that this is done. We have

$$T'_{m,a+x} = \lim_{r \to 0} \frac{T_m(x + r(x + a)) - T_m(x + rx)}{r} + \lim_{r \to 0} \frac{T_m(x + rx) - T_m(x)}{r} = T'_{m,a} + T_m(x),$$

since T_m is assured to be homogeneous. In this equation the existence of either side implies the existence of the other. And if $c_m \notin C_m$ we have

$$T'_{m,a+c_m} = \lim_{r \to 0} \frac{T_m(x + r(a + c_m)) - T_m(x)}{r} = \lim_{r \to 0} \frac{T_m(rc_m) + T_m(x + ra) - T_m(x)}{r} = T'_{m,a} + T_m(c_m).$$

The second last equality holds since T_m is linear on every $(m+1)$-dimensional subspace of B which contains C_m. Now let C_{m+1} be the $(m+1)$-dimensional subspace of B generated by C_m and x. And let D_{m+2} be an $(m+2)$–dimensional subspace of B generated by C_{m+1} and an element a, $a \in N$. Then we have $T_{m+1}(tx + c_m + va) = T_{m+1}(x - (1-t)x + c_m + va) = T_m(x) - T_m(1-t)x + T_m(c_m)) + v \cdot T'_{m,a} = T_m(tx) + T_m(c_m) + v \cdot T'_{m,a}$. This shows that T_{m+1} is linear on D_{m+2} and since N is dense on the boundary of the unit sphere in B, T_{m+1} is linear on every $(m+2)$-dimensional subspace of B which contains C_{m+1}. The lemma is proved.

LEMMA 4. *If for an n-dimensional Banach space B there exists a Lipschitz-embedding T of B into Hilbert space such that* $\| T \| \| T^{-1} \| \leqq K$ *then there exists an isomorphism V from B onto Euclidean n-space such that* $\| V \| \| V^{-1} \| \leqq K$.

PROOF. By considering a point x where T has a derivative in all directions of N (defined in the preceding lemma) we get a homogenuous embedding of B into Hilbert space with $\| T_0 \| \| T_0^{-1} \| \leqq K$ by defining

$$T_0(y) = \lim_{r \to 0} \frac{T(x + ry) - T(x)}{r}$$

if T has a derivative in the direction of y at x. We extend T_0 by continuity to all of B. Then the conditions of Lemma 4 are satisfied with $m = 0$. By applying Lemma 4 n $-$ 1 times and putting $T_{n-1} = V$ we get the desired isomorphism.

PROOF OF THEOREM 6.3.1. We now apply the following theorem by Dvoretsky [17]: A Banach space is isomorphic to a Hilbert space if and only if there is a real number K such that for every n and any two n-dimensional subspaces B_1 and B_2 of B there is an isomorphism $V: B_1 \to B_2$ with $\| V \| \| V^{-1} \| \leqq K$. If a Banach space is uniformly homeomorphic to a Hilbert space then Lemma 2 and Lemma 5 show that the conditions of Dvoretsky's theorem are satisfied and so the Banach space is isomorphic to a Hilbert space. The theorem is proved.

Theorem 6.2.1 and Theorem 6.3.1 give, since a Hilbert space has roundness 2

THEOREM 6.3.2. *If a locally bounded linear space is uniformly homeomorphic to a Hilbert space, then it is isomorphic to the Hilbert space.*

Since in the proofs of Lemma 4 and Lemma 5 above we could equally well work with local embeddings, these lemmas and the conclusion of the proof of Theorem 6.3.1 give

THEOREM 6.3.3. *If a Banach space is locally Lipschitz embeddable a Hilbert space, then it is isomorphic to a Hilbert space.*

Theofem 6.3.1 and the corollary of Theorem 5.1.5 give

THEOREM 6.3.4. *If a connected subgroup G of a metric linear space is locally Lipschitz equivalent to a Hilbert space, then G is a linear space which is isomorphic to the Hilbert space.*

We think that Theorem 6.3.3 becomes wrong if "locally Lipschitz embeddable" is changed to "uniformly embeddable". Theorem 6.3.4 becomes wrong if "locally Lipschitz equivalent" is changed to "Lipschitz embeddable in" as is shown by Example 5.2.1, and we think that it remains false even if we assume that G is a metric linear space.

The technique of Lemma 4 easily proves that if there is a local Lipschitz homeomorphism T between two finite-dimensional Banach spaces such that $\| T \| \| T^{-1} \| \leqq K$ then there is an isomorphism V between the spaces with $\| V \| \| V^{-1} \| \leqq K$. This result follows also directly from a theorem of Rademacher, which says that a Lipschitz mapping from R_0 to R is differentiable almost everywhere. However, the technique of Lemma 4 seems to be useful also when constructing isomorphisms between infinite-dimensional Banach spaces assumed to be Lipschitz equivalent but we have not worked out any details.

7. Structure theorems for commutative Banach groups

7.1. In this chapter we combine results from earlier sections to get some results on the structure of commutative groups. We have

THEOREM 7.1.1. *If for a commutative Banach group B with $p_B > 1$,*
(a) $\|(x_2 y - x_1 y)\| = o(\| x_2 - x_1 \|^{1/p_B})$ *uniformly in x_1, x_2 and y as $\|x_2 - x_1\| \to 0$,*
(b) *the group is uniformly dissipative, then the group is isomorphic to a Banach space, and if B is a Hilbert space the group is isomorphic to B.*

PROOF. Theorem 5.1.1 gives that the set of elements of the form y^2 is dense n B. Then Theorem 2.2.3 gives that the group is locally bounded linear space and then Theorem 6.2.1 and Theorem 6.3.1 complete the proof.

THEOREM 7.1.2. *If for a commutative local Banach group U, where the Banach space B is uniformly convex or has roundness > 1* (a) $\| x_2 y - x_1 y \| \leqq K \| x_2 - x_1 \|$ *for all x_1, x_2 and y in some neighbourhood of 0 for some K,*
(b) *there is a neighbourhood V of 0 such that $x \in V$, $y \in V$ and $x^2 = y^2 \Rightarrow x = y$,*

then the group is a local Banach space and if B is a Hilbert space then the group is a local Hilbert space.

PROOF. Since we have assumed uniqueness of square roots we get when applying Theorem 5.1.5 square roots on exactly half the distance to 0. Theorem 2.3.2 then shows that the group is a local Banach space and Theorem 6.3.1 gives the case when B is a Hilbert space.

It is a natural question whether the condition (b) in Theorem 7.1.2 can be removed.

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